ON VARIANTS OF THE HALTON SEQUENCE

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ABSTRACT. A generalisation of the classical Halton sequence $(\phi_\beta(n))_{n\in\mathbb{N}}$ has emerged in recent years based on β-adic expansions of elements of $[0,1)$. In the case where β is a natural number greater than 1 this reduces to the classical Halton sequence. In this paper we use ergodic theoretic methods, to prove the uniform distribution of the sequence $(\phi_\beta(k_j))_{j\in\mathbb{N}}$ for the sequence of integers $(k_j)_{j\geq 0}$ which is both Hartman uniformly distributed and good universal. This builds on earlier work of M. Hofer, M. R. Iaco and R. Tichy in the special case $k_j = j$ $(j = 0, 1, \ldots)$. Variants of this phenomenon are also studied.

1. Introduction

A standard problem in numerical analysis, is estimating the integral of a function, through knowledge if its value at a finite number of points $(x_n)_{n=1}^N$. This is known as Monti Carlo estimation in the case of stochastic sequences $(x_n)_{n=1}^N$ or Quasi-Monti Carlo estimation in the case of deterministic $(x_n)_{n=1}^N$. This is encapsulated in the famous Denjoy-Koksma inequality

$$\left|\frac{1}{N}\sum_{n=1}^{N} f(\{x_n\}) - \int_0^1 f(x)dx\right| \leq V(f)D(x_1, \ldots, x_N),$$

2010 Mathematics Subject Classification: 11K31, 40A05.
Keywords: Hartman uniformly distributed sequenes, Halton sequences and β-transformations.

where for a function f on $[0,1)$ with variation $V(f)$ and any finite set of points $\{x_1,\dots,x_N\}$ in $[0,1)$ with discrepancy

$$D_N = D(x_1,\dots,x_N) = \sup_{I\subseteq[0,1)} \left| \frac{1}{N}\#\{1\le n\le N : x_n\in I\} - |I| \right|.$$

Here the supremum is taken over all intervals $I\subseteq[0,1)$ closed on the left and open on the right and $|I|$ denotes the length of the interval I. In practical applications in fact the generalisation of the Denjoy Koksma inequality to higher dimensions is of more likely to be of practical use. Here the Hardy-Krause variation has to be substituted for $V(f)$. This does produce certain technical complications but these issues will not concern us in this paper. For clarity of exposition and brevity, we will confine ourselves to one dimension at this point. Evidently to estimate $\int_0^1 f(x)dx$, sufficiently precisely what is needed is a good bound for D_N and a serviceable bound for $V(f)$, which is usually straight forward. To be useful this sequence $\{x_1,\dots,x_N\}$ must be uniformly distributed modulo one, i.e.

$$\lim_{N\to\infty}\frac{1}{N}\#\{1\le n\le N : x_n\in I\} = |I|$$

for every interval $I\subseteq[0,1)$. The discrepancy D_N is nothing other than a quantitative measure of uniformity of distribution. In particular, the sequence $(x_n)_{n\ge1}$ is uniformly distributed modulo one if and only if $D_N\to0$ as $N\to\infty$. In a sense the faster D_N decays as a function of N, the better uniformly distributed the sequence $(x_n)_{n\ge1}$ is. One of the fundamental obstructions in nature in this subject is that there is a limit to how well distributed any sequence can be. This is encapsulated in the elementary inequality $D_N\ge\frac{1}{N}$ $(N=1,2,\dots)$ whose proof makes an elementary exercise. This opens the door to the deep subject of irregularities of distribution which address just what limitations there are to the uniformity of distribution of an arbitrary sequence, and the complementary problem of constructing sequences with discrepancy as small as possible. This latter issue is clearly central to the initial issue mentioned in this paper. Perhaps the most famous example of a low discrepancy sequence is the Van der Corput sequence. This is described as follows. Let $\mathbb{Z}_p$ denote the p-adic integers, for a rational prime p which is described as follows. Let $\mathbb{Q}_p$ denote the completion of the rational numbers $\mathbb{Q}$ with respect the absolute value $|.|_p$ defined for a rational of the form $x=p^\rho\frac{u}{v}$ with $\rho\in\mathbb{Z}$ and both integers u and v coprime to p by

$$|x|_p = \begin{cases} p^{-\rho} & \text{if } x\ne0\\ 0 & \text{if } x=0.\end{cases}$$

We then set $\mathbb{Z}_p=\{x\in\mathbb{Q}_p : |x|_p\le1\}$. One check readily that $\mathbb{Z}\subset\mathbb{Z}_p$ and one can show that the natural numbers are uniformly distributed in $\mathbb{Z}_p$. There are a variety of ways of showing this [KN]. One is to to use Birkhoff's pointwise ergodic theorem and the unique ergodicity of the the adding machine map defined $Tx = x + 1$ on $\mathbb{Z}_p$, which is just a dense group rotation on $\mathbb{Z}_p$. More on this later. A basic property of the p-adic numbers is that each element x of $\mathbb{Z}_p$ has a unique expansion of the form

$$x = \sum_{n=1}^{\infty} a_n p^n$$

where for each n in $\mathbb{N}$ we have $a_n \in \{0, 1, \ldots, p-1\}$. To each p there is a map $\phi_p : \mathbb{Z}_p \to [0,1)$ called the Monna map defined for $x = \sum_{n=1}^{\infty} a_n p^n$ to be $\phi_p(x) = \sum_{n=1}^{\infty} a_n p^{-n}$. The Monna map is bijective off the set of negative integers. The p-adic Van der Corput sequence is the sequence $(\phi_p(n))_{n\geq 1}$. Since the introduction of the Van der Corput sequence there has been an interest in variants of this construction. A first step is that we can in similar fashion define the b-adic numbers, the b-adic Monna map and the b-adic Van der Corput sequence for any $b > 1$. This is done via the base b expansion of a real number in $[0,1)$ and a group called the b-adic integers $\mathbb{Z}$. A sequence $(x_n)_{n\geq 1}$ is said to be uniform distribution on an arbitrary compact topological group G if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_G f(x)dx,$$

for all continuous functions f on G. Here dx denotes Haar measure on G. In the case $G = [0,1)$ (viewed as a group under addition) this definition can be shown to be equivalent to the definition given above. In the case when f is restricted to the set of characters of G, we get an equivalent characterisation called Weyl's criteria and is of interest because it is often the easiest formulation with which to prove the uniform distribution of a particular sequence. Given coprime integers $\{b_1, \ldots, b_s\}$ all greater than 1. The sequence $(\phi_{b_1}(n), \ldots, \phi_{b_s}(n))_{n\geq 1}$ is called Halton sequence and is uniformly distributed on $[0,1)^s$. It is also an example of a low discrepancy sequence in $[0,1)^s$, and as such of considerable interest in Monti Carlo estimation. There has also been a considerable interest in the distribution properties of the subsequences of this sequence. In resent years there has been an interest in analogues of the Van der Corput and Halton sequences where the role of the numbers $\{b_1, \ldots, b_s\}$ is taken by real numbers $\{\beta_1, \ldots, \beta_s\}$ with $\beta_i > 1$ for all $i = 1, 2, \ldots, s$. A dynamical approach to the distribution of these sequences hinted at above, is based on the ergodic theory of the Parr-Renyi beta transformations $T_i(x) = \{\beta_i x\} \bmod 1$ $(i = 1, 2, \ldots, s)$. In this paper we are interested in the distribution of subsequences and other of its variants, primarily by ergodic but also some analytic methods. The results of this paper are suggested by work of Robert Tichy and some of his co-authors.

2. Base Beta Halton Sequences

Let $(G_n)_{n\geq 0}$ be an increasing sequence of positive integers with $G_0 = 1$. Then every natural number n can be written

$$n = \sum_{k=0}^{\infty} g_k(n) G_k,$$

where $g_k(n) \in \{0, \ldots, [G_{k+1}/G_k]\}$ and $[x]$ denotes the integer part of x. This expansion (called the G- expansion) is unique and finite, provided that for every finite $K > 0$ that

$$n = \sum_{k=0}^{K-1} g_k(n) G_k < G_K. \tag{1}$$

We call g_k the k-digit of the G-expansion. The digits $(g_k)_{k\geq 0}$ can be calculated using the greedy algorithm and $G = (G_k)_{n\geq 0}$ is called an enumeration system. We denote by $\mathcal{K}_G$ the subset of sequences satisfying (1). The elements of $\mathcal{K}_G$ are called G-admissible. To extend the addition-by-1 map from $\mathbb{N}$ to $\mathcal{K}_G$ we introduce

$$\mathcal{K}_G^0 = \left\{ x \in \mathcal{K}_G : \exists M_x, \forall j \geq M_x, \sum_{k=0}^{j} g_k(n) G_k < G_{j+1} - 1 \right\} \subseteq \mathcal{K}_G.$$

Put $x_j = \sum_{k=0}^{j} g_k G_k$ and set

$$\tau(x) = (g_0(x_j + 1) \ldots g_j(x_j + 1) g_{j+1}(x) g_{j+2}(x) \ldots,$$

for every $x \in \mathcal{K}_G^0$ and $j \geq M_x$. This definition does not depend on the choice of $j \geq M_x$ and can be extended to $\mathcal{K}_G \backslash \mathcal{K}_G^0$ by setting $\tau(x) = 0 = (0)^{\infty}$. We have defined the G-odometer or G-adding machine .

In the sequel we restrict attention to enumeration systems where $G = (G_n)_{n\geq 0}$ is a linear recurrence i.e. we require in addition that for each positive integer n that

$$G_{n+d} = a_0 G_{n+d-1} + \ldots + a_{d-1} G_n.$$

To this linear recurrence we can associate the characteristic equation

$$x^d = a_0 x^{d-1} + \ldots + a_{d-1},$$

We further confine attention to enumeration systems, with a characteristic equation having a Pisot-Vijayragahavan number (PV) as a root. Note that this is always the case when

$$a_0 \geq a_1 \geq \ldots \geq a_{d-1} \geq 1.$$

W. Parry showed that under this hypothesis, the β-expansion of β is finite, i.e.

$$\beta = a_0 + \frac{a_1}{\beta} + \ldots + \frac{a_{d-1}}{\beta^{d-1}}, \tag{2}$$

where $a_0 = [\beta][P]$.

To enumeration systems, whose characteristic root β is a PV-number satisfying (2), a sum $\sum_{k=0}^{M} g_k G_k$ for finite M is the expansion of an integer if and only if the digits of the G-expansion satisfy

$$(g_k, g_{k-1}, \ldots, g_0, 0^\infty) < (a_0, a_1, \ldots, a_{d-1})^\infty,$$

for each k with $<$ denoting the lexicographic order. Representations $(g_k, \ldots, g_0)$ satisfying the condition are called admissible representations and thus belong to $\mathcal{K}_G$. Let Z denote a cylinder of length k for the dynamical system $(\mathcal{K}_G, \tau)$ viewed symbolically and let $F_{k,r} = \#\{n < G_{k+r} : (g_0(n), g_1(n) \ldots) \in Z\}$. We can define the measure μ on $\mathcal{K}_G$ by

$$\mu(Z) = \frac{F_{K,0}\beta^{d-1} + (F_{K,1} - a_0 F_{K,0}) + \ldots + (F_{K,d-1} - a_0 F_{K,d-2} - \ldots - a_{d-2}F_{K,0})}{\beta^K(\beta^{d-1} + \beta^{d-1} + \ldots + 1)}.$$

Note that if $(k_n)_{n\geq 0}$ is Hartman uniformly distributed and if we define

$$F(N, z) := \frac{1}{N}\sum_{n=0}^{N-1} z^{k_n}, \qquad (N = 1, 2, \cdots)$$

we have $F(N, 1) = 1$ for all $N \geq 1$ and if $z \neq 1$ we have $\lim_{N\to\infty} F(N, z) = 0$.

We say $(k_n)_{n\geq 0}$ is good universal if for each dynamical system $(X, \mathcal{B}, \mu, T)$ and each essentially bounded function on X the limit

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} f(T^{k_n}x),$$

exists μ everywhere.

We now turn to the definition of the Monna map ϕ_β for irrational bases β as follows. Let $n = \sum_{j\geq 0} g_j(n) G_j$ be the G-expansion of the positive integer n. Then define the

$$\phi_\beta(n) = \phi_\beta\left(\sum_{j\geq 0} g_j(n) G_j\right) = \sum_{j\geq 0} g_j(n)\beta^{-j}.$$

Furthermore the restriction to $\mathcal{K}_G^0$ has a well defined inverse. In this context the β-adic Halton sequence is given as $(\phi_\beta)_{n\geq 0} = (\phi_{\beta_1}(n), \ldots, \phi_{\beta_s}(n))_{n\geq 0}$, where $\beta = (\beta_1, \ldots, \beta_s)$, and β_i is the characteristic root of the G_i-expansion.

By the nature of its construction, as shown in Proposition 1.1, we see that $\phi_\beta(\mathbb{N}) \subset [0,1)$. Of course this implies nothing about its distribution. The following gives conditions for the density of $\phi_\beta(\mathbb{N})$ [HIT].

Proposition 1.1 *Let* $\mathbf{a} = (a_0, \ldots, a_{d-1})$ *where* $a_0, \ldots, a_{d-1} \geq 0$ *are the coefficients defining the enumeration system* G *and assume the* β *expansion of* β *are finite. Further more assume that there is no* $\mathbf{b} = (b_1, \ldots, b_{k-1})$ *with* $k < d$ *such that* β *is the characteristic root of the polynomial defined by* $\mathbf{b}$. *Then* $\phi_\beta(\mathbb{N}) \subset [0,1)$ *and* $\phi_\beta(\mathbb{N})$ *is not contained in* $[0,x)$, *for all* $x \in (0,1)$ *if and only if* $\mathbf{a}$ *can be written either as* $\mathbf{a} = (a_0, \ldots, a_0)$ *or as* $\mathbf{a} = (a_0, a_0 - 1 \ldots, a_0 - 1, a_0)$ *where* $a_0 > 0$.

In this paper we prove the following theorem.

Theorem 1: *Let* $G^1, \ldots, G^s$ *be enumeration systems defined by linear recurrences given as* $a^i_j = b_i$, $j = 0, \ldots, (d_i - 1), i = 1, \ldots, s$ *with pairwise coprime positive integers* b_i $i = 1, \ldots, s$. *Furthermore suppose* $\frac{\beta^i_i}{\beta^i_j} \notin \mathbb{Q}$. *Then if* $(k_j)_{j\geq 1}$ *is Harman uniformly distributed and* L^∞*-good universal, the sequence* $(\phi_\beta(k_j))_{j\geq 1}$ *is uniformly distributed on* $[0,1)^s$.

3. Hartman uniform distribution and unique ergodicity

Let $(X, \mathcal{B}, \mu)$ be a probability space and let $T : X \to X$ be a measurable map, that is also measure-preserving. That is, given $A \in \mathcal{B}$, we have $\mu(T^{-1}A) = \mu(A)$, where $T^{-1}A$ denotes the set $\{x \in X : Tx \in A\}$. In this paper we say a sequence $\mathbf{k} = (k_i)_{i=1}^\infty$ is *L^p-good universal* , if for each probability space $(X, \mathcal{B}, \mu)$, each measurable measure-preserving map T on it and all functions $f \in L^p(X, \mathcal{B}, \mu)$ we have the limit

$$\ell_{T,f}(x) := \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} f(T^{k_i}x),$$

existing almost everywhere with respect to μ.

Lemma 3.1. *Suppose* $(a_n)_{n=1}^\infty$ *is Hartman uniformly distributed on* $\mathbb{Z}$, *the dynamical system* $(X, \mathcal{B}, \mu, T)$ *is ergodic and* $(a_n)_{n\geq 1}$ *is* L^p*-good universal for* $p \in [1,2]$. *Then* $\ell_{T,f}(x)$ *exists and equals* $\int_X f d\mu$ μ *almost everywhere.*

Proof of Lemma 3.1 : First assume $f \in L^2$. Because $\left|\frac{1}{N}\sum_{n=1}^{N} f(T^{a_n}x)\right| \leq Mf(x)$ $(N = 1, 2, \ldots)$ and $Mf^2 \in L^1$, the dominated convergence theorem implies

$$g(x) = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(T^{a_n}x)$$

exists in norm. Our next order of business is to show that $g(Tx) = g(x)$. Let $Uf(x) = f(Tx)$. This is a unitary operator on L^2 as T is measure-preserving. Also let U^{-1} denote the L^2 adjoint of U. Recall that we say any sequence $(c_n)_{n\in\mathbb{Z}}$ is positive definite if given a bi-sequence of complex numbers $(z_n)_{n\in\mathbb{Z}}$, only finitely many of whose terms are non-zero, we have $\sum_{n,m\in\mathbb{Z}} c_{n-m} z_n \overline{z_m} \geq 0$. Here $\overline{z}$ is the conjugate of the complex number z. Let $\langle f, g\rangle = \int_X f\overline{g}d\mu$ (i.e. the standard inner product on L^2). Notice that $(\langle U^n f, f\rangle)_{n\in\mathbb{Z}}$ is positive definite. Recall that the Bochner-Herglotz theorem [Kt] says that there is a measure ω_f on $\mathbb{T}$ such that

$$\langle U^n f, f\rangle = \int_{\mathbb{T}} z^n d\omega_f(z). \qquad (n \in \mathbb{Z})$$

This tells us that

$$\left\|\frac{1}{N}\sum_{n=1}^{N} f(T^{a_n+1}x) - \frac{1}{N}\sum_{n=1}^{N} f(T^{a_n}x)\right\|_2^2$$

$$= \int_{\mathbb{T}} (2 - z - z^{-1}) \left|\frac{1}{N}\sum_{n=1}^{N} z^{a_n}\right|^2 d\omega_f(z)$$

using the parametrization $z = e^{2\pi i\theta}$ for $\theta \in [0, 1)$, this is

$$= 4\int_{\mathbb{T}} \sin^2\left(\frac{\theta}{2}\right) \left|\frac{1}{N}\sum_{n=1}^{N} z^{a_n}\right|^2 d\omega_f(z).$$

Using the fact that $\sin\frac{\theta}{2} = 0$ if $\theta = 0$ and the fact that $(a_n)_{n\geq 1}$ is Hartman uniformly distributed we see that $g(Tx) = g(x)$. As observed earlier, by [CFS] p. 14 if T is ergodic and $g(Tx) = g(x)$ for measurable g, then $g(x)$ must be $\int_X f d\mu$. The same observation extends to L^p as L^2 is dense in L^p.

All we have to do now is show the pointwise limit is the same as the norm limit, i.e. that $\overline{f}(x) = g(x) = \int_X f d\mu$. We consider a sequence of natural numbers $(N_t)_{t\geq 1}$ such that

$$\left\|\frac{1}{N_t}\sum_{n=1}^{N_t} f(T^{a_n}x) - \int_X f(x)d\mu\right\|_p \leq \frac{1}{t}.$$

Thus

$$\sum_{t=1}^{\infty}\int_X\left|\frac{1}{N_t}\sum_{n=1}^{N_t} f(T^{a_n}x) - \int_X f(x)d\mu\right|^p d\mu < \infty.$$

Fatou's lemma tells us that

$$\int_X\left(\sum_{t=1}^{\infty}\left|\frac{1}{N_t}\sum_{n=1}^{N_t} f(T^{a_n}x) - \int_X f(x)d\mu\right|^p\right) d\mu < \infty,$$

which implies that

$$\sum_{t=1}^{\infty}\left|\frac{1}{N_t}\sum_{n=1}^{N_t} f(T^{a_n}x) - \int_X f(x)d\mu\right|^p < \infty,$$

almost everywhere. This means that

$$\left|\frac{1}{N_t}\sum_{n=1}^{N_t} f(T^{a_n}x) - \int_X f(x)d\mu\right| = o(1),$$

μ almost everywhere. As $(a_n)_{n\geq 1}$ is L^p-good universal we must have $\overline{f}(x) = \int_X f(x)d\mu$ μ almost everywhere with respect to μ. □

We have the following Lemma, which in the case $k_n = n(n = 1, 2, \ldots)$.

Lemma 3.2. *Suppose $(k_n)_{n\geq 0}$ is Hartman uniformly distributed and L^2-good universal. Let T be a continuous map of a compact metrizable space X. Also let μ denote a measure defined on a σ-algebra $\mathcal{B}$ of subsets of X. The following statements are equivalent :*

a) the transformation $(X, \mathcal{B}, \mu, T)$ is uniquely ergodic ;

b) for each continuous function f defined on X there is a constant C_f independent of x such that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1} f(T^{k_n}x) = C_f,$$

c) for each continuous function f defined on X there is a constant C_f independent of x such that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1} f(T^{k_n}x) = C_f,$$

uniformly on X;

and

d) whenever f is in $C(X)$ (the space of continuous functions on X)

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1} f(T^{k_n}x) = \int_X f d\mu,$$

pointwise on X, i.e. for all $x \in X$.

Proof. Evidently c) implies b). We next consider the proof that d) implies a). Let

$$S_N f(x) = \frac{1}{N}\sum_{n=0}^{N-1} f(T^{k_n}x). \qquad (N = 1, 2, \ldots)$$

For ν in $M(X,T)$, by the dominated convergence theorem we have

$$\lim_{N\to\infty}\int_X S_N f(x)d\nu(x) = \int_X f d\nu = \int_X f d\mu.$$

This holds for all f in $C(X)$ and hence by the Riesz representation theorem we have $\nu = \mu$, as required.

We now prove b) implies d). Set

$$k(f) = \lim_{N\to\infty}\frac{1}{N}\sum_{j=0}^{N-1} f(T^{k_j}x).$$

Observe that k is a linear operator and is continuous since

$$\left|\frac{1}{N}\sum_{j=0}^{N-1} f(T^{k_j}x)\right| \le \|f\|.$$

Also as $k(1) = 1$ and $k(f) \ge 0$ if $f \ge 0$ we have $k(f) \ge 0$. Thus by the Riesz-Representation theorem $k(f) = \int_X f d\mu$ with respect to a Borel probability measure μ. Also note $k(f \circ T) = k(f)$ so $\int_X f \circ T d\mu = \int_X f d\mu$. Thus $\mu \in M(X,T)$.

We now show how a) implies c). Suppose b) does not hold. Then there exists an $\epsilon > 0$, a function g in $C(X)$ and a sequence $(x_n)_{n=1}^{\infty}$ in X such that

$$|S_n g(x_n) - \int_X g d\mu| > \epsilon.$$

Let

$$\mu_n = \frac{1}{n}\sum_{i=0}^{n-1}\delta_{T^{k_i}(x_n)}, \qquad (n = 1, 2, \ldots)$$

where δ_y denotes the delta function based at y. This means

$$\left|\int_X g d\mu_N - \int_X g d\mu\right| \geq \epsilon.$$

As the set of measures $M(X)$ is compact, we can pick a subsequence (μ_{n_j}) convergent to μ_∞, say. We want to prove μ_∞ is T-invariant. Then unique ergodicity would imply μ_∞ and μ are different, so a) would imply b) as required. Note

$$\int_X g d\mu_\infty - \int_X g \circ T d\mu_\infty = \int_X (1-T) g d\mu_\infty$$

$$= \lim_{N\to\infty} \int_X (1-T) g d\mu_N$$

$$= \lim_{N\to\infty} \int_X (1-T) g d\left(\frac{1}{N}\sum_{i=0}^{N-1} \delta_{T^{k_i}(x_{n_j})}\right),$$

$$= \lim_{N\to\infty} \int_X (1-T)\left(\frac{1}{N}\sum_{i=0}^{N-1} T^{k_i}\right) g d\delta_{(x_{n_j})},$$

This means that

$$\left|\int_X (1-T) g d\mu_\infty\right| \leq \lim_{N\to\infty} \int_X \left|(1-T)\left(\frac{1}{N}\sum_{i=0}^{N-1} T^{k_i}\right) g\right| d\delta_{(x_{n_j})}.$$

Integrating both sides of this inequality with respect to μ and noting the left hand side is a constant, we have

$$\left|\int_X (1-T) g d\mu_\infty\right| \leq \int_X \left(\lim_{N\to\infty} \int_X \left|(1-T)\left(\frac{1}{N}\sum_{i=0}^{N-1} T^{k_i}\right) g\right| d\delta_{(x_{n_j})}\right) d\mu.$$

Using dominated convergence, this is

$$\left|\int_X (1-T) g d\mu_\infty\right| \leq \int_X \left(\lim_{N\to\infty} \int_X \left|(1-T)\left(\frac{1}{N}\sum_{i=0}^{N-1} T^{k_i}\right) g\right| d\delta_{(x_{n_j})}\right) d\mu.$$

and using Fubini's theorem and observing that $\int_X d\delta(x_n) = 1$ this is

$$\leq \lim_{N\to\infty} \int_X \left|(1-T)\left(\frac{1}{N}\sum_{i=0}^{N-1} T^{k_i}\right) g\right| d\mu.$$

Using Cauchy's inequality this is

$$\leq \lim_{N\to\infty} \left\|(1-T)\left(\frac{1}{N}\sum_{i=0}^{N-1} T^{k_i}\right) g\right\|_2.$$

Using the Bochner-Herglotz theorem, there is a spectral measure ω_g attached to the function g and implicitly to the map T this is

$$\leq \lim_{N\to\infty}\left(\int_{\mathbb{T}}\left|(2-z-z^2)\left(\frac{1}{N}\sum_{i=0}^{N-1} z^{k_i}\right)\right|^2 d\omega_g\right)^{\frac{1}{2}}.$$

We write $z=e^{2\pi i\theta}$ on $\mathbb{T}$, then this is

$$= 4\lim_{N\to\infty}\left(\int_{\mathbb{T}}\left|\sin^2\left(\frac{\theta}{2}\right)\left(\frac{1}{N}\sum_{i=0}^{N-1} z^{k_i}\right)\right|^2 d\omega_g\right)^{\frac{1}{2}}.$$

When $\theta=0$ the inner integrand is zero. When $\theta\neq 0$ the integrand tends to zero as N tends to ∞. Thus $\int_X g d\mu_\infty = \int_X g\circ T d\mu_\infty$. Thus μ_∞ is in $M(X,T)$, and since

$$\left|\int_X g d\mu_\infty - \int_X g d\mu\right| \geq \epsilon,$$

a) implies b) as required. $\square$

4. Examples of Hartmann Uniformly distributed and good universal sequences

The following is a list of constructions of Hartman uniformly distributed sequences. The first five are also examples of L^p-good universal sequences for some $p\geq 1$. The other examples appear in [**N1**].

1. The sequence $(n)_{n=1}^{\infty}$ is L^1-good universal. This is Birkhoff's pointwise ergodic theorem.

2. Denote by $[y]$ the integer part of real number y. Set $k_n=[g(n)]$ $(n=1,\ldots)$ where $g:[1,\infty)\to[1,\infty)$ is a differentiable function whose derivation increases with its argument. Let A_n denote the cardinality of the set $\{n : k_n\leq n\}$ and suppose for some function $a:[1,\infty)\to[1,\infty)$ increasing to infinity as its argument does, that we set

$$b_M = \sup_{\{z\}\in[\frac{1}{a(M)},\frac{1}{2})}\left|\sum_{n:a_n\leq M} e(zk_n)\right|.$$

(Here $e(x) = e^{2\pi ix}$ for real x.) Suppose also for some decreasing function $c : [1,\infty) \to [1,\infty)$ and some positive constant $C > 0$ that

$$\frac{b(M) + A_{[a(M)]} + \frac{M}{a(M)}}{A_M} \leq Cc(M).$$

Then if we have

$$\sum_{s=1}^{\infty} c(\theta^s) < \infty$$

for $\theta > 1$ we say that $\underline{k} = (k_n)_{n=1}^{\infty}$ satisfies conditions H [**N2**].

Specific sequences of integers that satisfy conditions H include $k_n = [g(n)]$ $(n = 1, 2, \ldots)$ where

I. $g(n) = n^{\omega}$ if $\omega > 1$ and $\omega \notin \mathbb{N}$.

II. $g(n) = e^{\log^{\gamma} n}$ for $\gamma \in (1, \frac{3}{2})$.

III. $g(n) = P(n) = b_k n^k + \ldots + b_1 n + b_0$ for $b_k, \ldots, b_1$ not all rational multiplies of the same real number.

IV. Hardy Fields: By a Hardy Field we mean a closed subfield (under differentiation), of the ring of germs at $+\infty$ of continuous real valued functions with addition and multiplication taken to be pointwise. Let L denote the union of all Hardy fields. If $(k_n)_{n=1}^{\infty} = ([a(n)])_{n=1}^{\infty}$, where a satisfies the following conditions:

$$a \in L;$$

for some $k \in \mathbb{Z}$, $k \geq 2$

$$\lim_{x\to\infty} \frac{a(x)}{x^{k-1}} = \infty \text{ and } \lim_{x\to\infty} \frac{a(x)}{x^k} = 0;$$

then $(k_n)_{n=1}^{\infty}$ satisfies condition H. This example is observed in [**BKQW**].

3. A random example: (i) Suppose $S = (n_k)_{n=1}^{\infty} \subseteq \mathbb{N}$ is a strictly increasing sequence of natural numbers. By identifying S with its characteristic function I_S we may view it as a point in $\Lambda = \{0,1\}^{\mathbb{N}}$ the set of maps from $\mathbb{N}$ to $\{0,1\}$. We may endow Λ with a probability measure by viewing it as a Cartesian product $\Lambda = \prod_{n=1}^{\infty} X_n$ where for each natural number n we have $X_n = \{0,1\}$ and specify the probability π_n on X_n by $\pi_n(\{1\}) = q_n$ with $0 \leq q_n \leq 1$ and $\pi_n(\{0\}) = 1 - q_n$ such that $\lim_{n\to\infty} q_n n = \infty$. The desired probability measure on Λ is the corresponding product measure $\pi = \prod_{n=1}^{\infty} \pi_n$. The underlying σ-algebra β is that generated by the "cylinders"

$$\{\lambda = (\lambda_n)_{n=1}^{\infty} \in \Lambda : \lambda_{i_1} = \alpha_{i_1}, \ldots, \lambda_{i_r} = \alpha_{i_r}\}$$

for all possible choices of $i_1, \ldots, i_r$ and $\alpha_{i_1}, \ldots, \alpha_{i_r}$. Let $(k_n)_{n=1}^{\infty}$ be almost any point in Λ with respect to the measure π [**Bo1**].

4. Block sequences: If $(k_n)_{n\geq 1} = \cup_{k=1}^{\infty}[d_k, e_e]$ ordered by absolute value for disjoint $([d_k, e_k])_{k\geq 1}$ with $d_{k-1} = O(e_k)$ as k tends to infinity. Note that this allows the possibility that $(k_n)_{n\geq 1}$ is zero density. This example is an immediate consequence of A. A. Templeman's semigroup ergodic theorem [**T**, p218] .

5. Random perturbation of good sequences: Suppose $(k_n)_{n\geq 1}$ is a L^p-good universal sequence of integers that is also Hartman uniformly distributed. Suppose $\theta = \{\theta_n, n \geq 1\}$ denotes a sequence of $\mathbb{N}$-valued independent, identically distributed random variables with basic probability space $(\Omega, \mathcal{A}, \mathcal{P})$, and a $\mathcal{P}$-complete σ-field $\mathcal{A}$. We assume that there exist $0 < \alpha < 1$ and $B > 1/\alpha$, such that

$$k_n = O(e^{n^{\alpha}}),$$

and the if $\mathbb{E}$ denotes expectation with respect to the basic probability space $(\Omega, \mathcal{A}, \mathcal{P})$ we have

$$\mathbb{E} \log_{+}^{B} |\theta_1| < \infty.$$

Then $(k_n+\theta_n(\omega))_{n\geq 1}$ is L^p-good universal and Hartman uniformly distributed [**NW**].

6. $k_n = [P(n)]$ $(n = 1, 2, \ldots)$ where if $P(x) = a_k x^k + \cdots + a_1 x + a_0$ the numbers $a_k, \ldots, a_1$ are not all rational multiples of the same real number;

7. $k_n = [P(p_n)]$ $(n = 1, 2, \ldots)$ where $(p_n)_{n=1}^{\infty}$ denotes the sequence of rational primes and $P(z)$ is as in 6;

8. $k_n = [f(n)]$ $(n = 1, 2, \ldots)$ where $f(z)$ denotes a non-polynomial entire function which is real on the real numbers and such that $|f(z)| \ll e^{(logz)^{\alpha}}$ with $\alpha < \frac{4}{3}$;

9. $k_n = [f(p_n)]$ $(n = 1, 2, \ldots)$ where $f(z)$ is as in 8 and p_n denotes the n^{th} rational prime;

10. $k_n = [a_n cos(a_n x)]$ $(n = 1, 2, \ldots)$ for a strictly increasing sequence of integers $(a_n)_{n=1}^{\infty}$ and almost all x with respect to Lebesgue measure;

11. $k_n = [a_n cos(a_n x)]$ $(n = 1, 2, \cdots)$ for a strictly increasing sequence of integers $(a_n)_{n=1}^{\infty}$ such that $a_n \ll n^p$ and $p > 1$ and all x outside a set of Hausdorff dimension not greater than $1 - \frac{1}{4p + \frac{1}{2}}$;

12. $k_n = [g_n(x)]$ $(n = 1, 2, \cdots)$ for almost all x with respect to Lebesgue measure in $[a, b]$ where $(g_n(x))_{n=1}^{\infty}$ is a sequence of continuously differentiable functions defined on $[a, b]$ satisfying the following hypothesis. For each pair of distinct natural numbers m and n we have

(a) $g_n'(x) - g_m'(x)$ is monotonic on $[a, b]$ and

(b) there is an absolute constant λ such that

$$|g_n'(x) - g_m'(x)| \ \geq \ \lambda \ > \ 0.$$

13. $k_n = [g_n(x)]$ $(n = 1, 2, \cdots)$ for all x lying outside a set of Hausdorff dimension at most $1 - \frac{1}{p}$ in $[a, b]$ where $(g_n(x))_{n=1}^{\infty}$ is a sequence of continuously differentiable functions defined on $[a, b]$ satifying the hypothesis (a), (b) of 12 and in addition

(c) for all x in $[a, b]$ we have

$$\sup_{x \in [a,b]} |g_n'(x)| \ll n^p$$

for some $p > 1$ and with an implied constant independent of x

and

(d) for each pair of distinct positive integers m and n the function

$$\frac{g_n'(x) g_m'(x)}{g_m'(x) - g_n'(x)}$$

is monotonic on $[a, b]$.

5. Polynomial and polynomial in prime sequences

We consider a different framework. To fix ideas we begin with some basic information about the $\mathbf{a}$-adic integers. Let $\mathbf{a} = (a_n)_{n=0}^{\infty}$ be a sequence of rational integers greater than one. Using the ideas of [HR §10] we define the $\mathbf{a}$-adic integers $\mathbb{Z}_{\mathbf{a}}$ to be the set of infinite sequences $(x_n)_{n=0}^{\infty}$ in $\Pi_{n=1}^{\infty}\{0, 1, \ldots, a_n - 1\}$.

For $\underline{x} = (x_n)_{n=0}^{\infty}$ and $\underline{y} = (y_n)_{n=0}^{\infty}$ let $\underline{z} = (z_n)_{n=0}^{\infty}$ be defined as follows. Write $x_0 + y_0 = t_0 a_0 + z_0$, where $z_0 \in \{0, 1, ..., a_0 - 1\}$ and t_0 is a rational integer. Suppose $z_0, \ldots, z_k$ and $t_0, \ldots, t_k$ have been defined. Then write $x_{k+1} + y_{k+1} + t_k = t_{k+1} a_{k+1} + z_{k+1}$, where $z_{k+1} \in \{0, 1, ..., a_{k+1} - 1\}$ and t_{k+1} is a rational integer. We have thus inductively defined the sequence $\underline{z} = (z_n)_{n=0}^{\infty}$, which we deem to be $\underline{x} + \underline{y}$. The binary operation $+$ which we call addition makes $\mathbb{Z}_{\mathbf{a}}$ an Abelian group.

For each non-negative integer k let

$$\Lambda_k = \{x \in \mathbb{Z}_{\mathbf{a}} : x_n = 0 \ if \ n < k\}.$$

These sets form a basis at $\underline{0} = (0, 0, \ldots)$ for a topology on $\mathbb{Z}_{\mathbf{a}}$. With respect to this topology $\mathbb{Z}_{\mathbf{a}}$ is compact and the group operations are continuous making $\mathbb{Z}_{\mathbf{a}}$ a compact Abelian topological group. A second binary operation called multiplication, denoted by $\times$ and compatible with addition is defined as follows. Let $\underline{u} = (1, 0, 0, \ldots)$. Note that $(n\underline{u})_{n=0}^{\infty}$ is dense in $\mathbb{Z}_{\mathbf{a}}$. First on the set $\{n\underline{u} : n = 1, 2, \ldots\}$ define $k_1\underline{u} \times k_2\underline{u}$ to be $k_1k_2\underline{u}$. Deeming multiplication to be continuous on $\mathbb{Z}_{\mathbf{a}}$ defines it off the set $\{n\underline{u} : n = 0, 1, \ldots\}$. The binary operations addition and multiplication makes $\mathbb{Z}_{\mathbf{a}}$ a topological ring.

For each non-negative integer n, let $\lambda_n(A)$ denote the measure on the finite set $\{0, 1, \ldots, a_n - 1\}$ given by $\lambda_n(A) = \mathrm{card}(A)/a_n$. Haar measure is the corresponding product measure on $\mathbb{Z}_{\mathbf{a}}$.

The dual group to $\mathbb{Z}_{\mathbf{a}}$, which we denote $\hat{\mathbb{Z}}_{\mathbf{a}}$ consists of all rationals $t = \frac{\ell}{A_r}$ where $A_r = a_0 \cdots a_r$ and $0 \leq \ell \leq A_r$ for some non-negative integer r. To evaluate a character χ_t at $\underline{x}$ in $\mathbb{Z}_{\mathbf{a}}$ we write

$$\chi_t(\underline{x}) = e\left(\frac{\ell}{A_r}(x_0 + a_0x_1 + \cdots + a_0 \cdots a_{r-1}x_r)\right),$$

where as usual, for a real number x, $e(x)$ denotes $e^{2\pi ix}$.

For a sequence $(x_n)_{n=0}^{\infty}$ in $\mathbb{Z}_{\mathbf{a}}$, let $A(E; N)$ denote the number of elements of the set $\{x_0, \ldots, x_{n-1}\}$ that belong to E. If for every set E belonging to the algebra generated by the sets Λ_k $(k = 0, 1, \ldots)$ and their translates, the limit

$$\mu(E) = \lim_{N\to\infty} \frac{A(E; N)}{N}$$

exists we say $(x_n)_{n=0}^{\infty}$ is asymptotically distributed on $\mathbb{Z}_{\mathbf{a}}$ with distribution μ. If μ coincides with Haar measure λ, we say $(x_n)_{n=0}^{\infty}$ is uniformly distributed.

There are sequences that are good universal though not Hartman uniformly distributed. Of particular interest are the sequences $k_j = \rho(n)$ and $\rho(p_j)$ $(j = 1, 2 \ldots)$ where ρ is a polynomial mapping the natural numbers to themselves and p_j is the j^{th} prime numbers. That these sequences are not in general Hartman uniformly distributed is evident from the observation that a square is never congruent to a three modulo four. It is natural to ask if whether either of the sequences $(\phi_\beta(\rho(n))_{n\geq 0}$ or $(\phi_\beta(\rho(p_n))_{n\geq 0}$ is uniformly distributed on $[0, 1)^s$. The answer is not in general as we shall show in this section though it is possible to say something about their distribution on $[0, 1)^s$. To show that either $(\phi_\beta(\rho(n))_{n\geq 0}$ or $(\phi_\beta(\rho(p_n))_{n\geq 0}$ are not uniformly distributed it is sufficient to show there exist continuous functions $f : [0, 1)^s \to \mathbb{C}$ such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} f(\phi_\beta(k_j)) = \int_{[0,1)^s} f d\lambda,$$

is false in the case that either $k_j = \rho(j)$ or $k_j = \rho(p_j)$. We can also say something about what happens instead. To do this we need to introduce an appropriate class of functions f. The map $\phi_{\mathbf{a}} : \mathbb{Z}_{\mathbf{a}} \to [0,1)$ is described for $\underline{x} = (x_n)_{n\geq 0} \in \mathbb{Z}_{\mathbf{a}}$ by $\phi_{\mathbf{a}}(\underline{x}) = \sum_{n\geq 0} \frac{x_n}{a_0 \ldots a_n}$. We identify the natural number n with the element $n\underline{u}$ of $\mathbb{Z}_{\mathbf{a}}$ and have therefore defined $\phi_{\mathbf{a}}(n)$. We shall refer to $\phi_{\mathbf{a}}$ as the Monna map. We call the maximal subset of $\mathbb{Z}_{\mathbf{a}}$ on which $\phi_{\mathbf{a}}$ is injective its *regular set.* We call the sequence $\mathbf{a}$ *useful* if the regular set contains all natural numbers. In the case $a_i = b$ for each natural number $b > 1$, the sequence $\mathbf{a}$ is checked to be useful and the sequence $(\phi_{\mathbf{a}}(n))_{n\geq 0}$ coincides with the base b Van der Corput sequence. One checks readily that on the regular set, the map $\phi_{\mathbf{a}}$ is a bijection and that the image of a uniformly distributed sequence on $\mathbb{Z}_{\mathbf{a}}$ is uniformly distributed on $[0,1)$. We denote the inverse of $\phi_{\mathbf{a}}$ on the regular set to be $\phi_{\mathbf{a}}^{+}$. Also more generally if a sequence is asymptotically distributed on $\mathbb{Z}_{\mathbf{a}}$ with respect to a measure ρ the image sequence is asymptotically distributed in $[0,1)$ with respect to the push forward of the measure ρ onto $[0,1)$. Define $\chi'_k : \mathbb{Z}_{\mathbf{a}} \to \{c \in \mathbb{C} : |c| = 1\}$ by

$$\chi'_k\left(\sum_{j\geq 0} z_j b^j\right) = e(\phi_{\mathbf{a}}(k)(z_0 + z_1 b \ldots)). \qquad (k = 1, 2 \ldots)$$

Note this is an equivalent way to describe an arbitrary character of the group $\mathbb{Z}_{\mathbf{a}}$. We can also lift each character χ'_k to a function on $[0,1)$ as follows. For $k \in \mathbb{N}$, let $\gamma_k : [0,1) \to \{c \in \mathbb{C} : |c| = 1\}$, defined by $\gamma_k(x) = \chi_k(\phi_{\mathbf{a}}^{+}(x))$ where $\phi_{\mathbf{a}}^{+}$ is given explicitly by

$$\phi_{\mathbf{a}}^{+}\left(\sum_{j\geq 0} x_j (a_0 \ldots a_j)^{-1}\right) = \sum_{j\geq 0} x_j a_0 \ldots a_{j-1},$$

for the set of regular points in $\mathbb{Z}_b$. Let $\Gamma_b = \{\gamma_k : k \in \mathbb{N}_0\}$. We readily see $\int_{[0,1)} \gamma_k d\lambda = 0$ where λ denotes Lebesgue measure. Note γ_k is an orthogonal system because

$$\int_{[0,1)} \gamma_k \overline{\gamma_l} d\lambda = 0, \qquad \forall\ k \neq l.$$

Now let $\mathbf{b} = (\mathbf{a}_1, \ldots, \mathbf{a}_s)$ be a vector of not necessarily distinct sequences $\mathbf{a}_i$ with $a_{i,j} \geq 2$ $(i = 1, 2, \ldots s)$ where $\mathbf{a}_i = (a_{i,j})_{j\geq 0}$, let $x = (x_1, \ldots, x_s) \in [0,1)^s$ and let $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$. We define

$$\phi_{\mathbf{b}}(\mathbf{k}) = (\phi_{\mathbf{a}_1}, \ldots, \phi_{\mathbf{a}_s}(k_s)).$$

We also define

$$\phi_{\mathbf{b}}^{+}(\mathbf{k}) = (\phi_{\mathbf{a}_1}^{+}, \ldots, \phi_{\mathbf{a}_s}^{+}(k_s)).$$

Now set $\gamma_{\mathbf{k}}(x) = \Pi_{i=1}^{s}\gamma_{i,k_i}(x_i)$ where $\gamma_{i,k_i} \in \Gamma_{\mathbf{a}_i}$, $1 \leq i \leq s$, and $\Gamma_{\mathbf{k}} = \{\gamma_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}_0^s\}$. Furthermore, for $n \in \mathbb{N}_0$, we write $\phi_{\mathbf{b}}(n) = (\phi_{\mathbf{a}_1}, (n), \ldots, \phi_{\mathbf{a}_s}(n))$. It is possible to show that $\Gamma_{\mathbf{b}}$ is an orthogonal basis for $L^2([0,1)^s)$. For $f \in L^1([0,1)^s)$, its $\mathbf{k}^{th}$-Fourier coefficient is given with respect to $\Gamma_{\mathbf{k}}$ by $\hat{f}(\mathbf{k}) = \int_{[0,1)^s} f\overline{\gamma}_{\mathbf{k}} d\lambda_s$.

Lemma 5.1: *The sequence $(x_n)_{n=0}^{\infty}$ in $\mathbb{Z}_{\mathbf{a}_1} \times \ldots \times \mathbb{Z}_{\mathbf{a}_s}$ is asymptotically distributed if and only if for each χ_t in $\hat{\mathbb{Z}}_{\mathbf{a}_1} \times \ldots \times \hat{\mathbb{Z}}_{\mathbf{a}_s}$*

$$c_t = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N} \chi_t(x_n)$$

exists. Also let $\Lambda_{\mathbf{r}}$ is the set of points in $\mathbb{Z}_{\mathbf{a}_1} \times \ldots \times \mathbb{Z}_{\mathbf{a}_s}$, whose first r_i digits in the i-coordinate $(i = 1, 2, \ldots, s)$ are zero and let

$$\mu_{\mathbf{r}}(\alpha) = \lim_{N\to\infty} \frac{A(\alpha + \Lambda_{\mathbf{r}}; N)}{N}.$$

Then if $A_{r_i,j} = a_{i,1} \ldots a_{i,r_i}$ $(i = 1, 2, \ldots, s)$ we have

$$c_t = \sum_{j_1=0}^{A_{r_1,1-1}} \cdots \sum_{j_s=0}^{A_{r_s,s}-1} \mu_r(j)e(t.\underline{j})$$

for each $\mathbf{r}$ with $r_i \geq 1$ for each $i = 1, 2, \ldots s$. Also if

$$Z_r = \left\{\left(\frac{l_1}{A_{u_1}}, \ldots, \frac{l_s}{A_{u_s}}\right) \in \hat{\mathbb{Z}}_{\mathbf{a}_1} \times \ldots \times \hat{\mathbb{Z}}_{\mathbf{a}_s} : 0 \leq u_i \leq r_i; (1 \leq i \leq s)\right\}$$

then

$$\mu_{\mathbf{r}}(\alpha) = \frac{1}{|Z_{\mathbf{r}}|} \sum_{t\in Z_{\mathbf{r}}} c_t e(-t.\alpha).$$

P r o o f. For each s-tuple of rational integers $\underline{j}$ an element y of $\mathbb{Z}_{\mathbf{a}_1} \times \ldots \times \mathbb{Z}_{\mathbf{a}_s}$ being in $\underline{j} + \Lambda_k$ implies that $e(t.y) = e(t.\underline{j})$ so for any sequence $(y_n)_{n=0}^{\infty}$ we have

$$\frac{1}{N}\sum_{n=0}^{N-1} e(t.y_n) = \sum_{j_1=0}^{A_{r_1,1}-1-1} \cdots \sum_{j_s=0}^{A_{r_s,s}-1} e(\underline{j}.t)N^{-1}|\{y_n \in \underline{j} + \Lambda_r : 1 \leq n \leq N\}|.$$

Therefore if $(y_n)_{n=0}^{\infty}$ is asymptotically distributed

$$c_t = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \chi_t(y_n)$$

exists. On the other hand letting

$$B_N \;=\; \sum_{t\in Z\mathbf{r}} e(-t.\alpha)\sum_{n=0}^{N-1} e(t.y_n) \;=\; \sum_{n=0}^{N-1}\sum_{t\in Z_{\mathbf{r}}} e(t.(y_n-\alpha)).$$

If y_n is in $\underline{j} + \Lambda_k$ this sum is $|Z_{\mathbf{r}}|\{y_n \in \underline{j} + \Lambda_{\mathbf{r}} : 1 \le n \le N\}|$. This means that,

$$\lim_{N\to\infty}\frac{1}{N}|\{y_n \in \underline{j} + \Lambda_{\mathbf{r}} : 1 \le n \le N\}| \;=\; \frac{1}{|Z_{\mathbf{r}}|}\sum_{t\in Z_{\mathbf{r}}} e(-t.j)c_t,$$

as required. □

For positive integer D and a polynomial η of the same degree (k say) as ρ define $G^{(1)}_{D,\eta} : \hat{\mathbb{Z}}_{\mathbf{a}_1}\times\ldots\times\hat{\mathbb{Z}}_{\mathbf{a}_s} \to \mathbf{C}$ by

$$G^{(1)}_{D,\eta}(\chi_t) = \frac{\chi_t(\alpha_0)}{D}\sum_{m=1}^{D_r} e^{2\pi i\frac{\eta(m)}{D}},$$

for all t in $\hat{\mathbb{Z}}_{a_1}\times\ldots\times\hat{\mathbb{Z}}_{\mathbf{a}_s}$. Here the positive integer D is determined by t, and the rational coefficients of ρ. Also consider $G^{(2)}_{D,\eta} : \hat{\mathbb{Z}}_{\mathbf{a}_1}\times\ldots\times\hat{\mathbb{Z}}_{\mathbf{a}_s} \to \mathbf{C}$ defined by

$$G^{(2)}_{D,\eta}(\chi_t) = \frac{\chi_t(\alpha_0)}{\phi(D)}\sum_{\substack{m=1\\(D,m)=1}}^{D} e^{2\pi i\frac{\gamma(m)}{D}},$$

where ϕ denotes Euler's totient function.

We recall the definition of the Von Mangolt function

$$\Lambda(n) = \begin{cases}\log_e(p) & \text{if } n = p^l \text{ for prime } p \text{ and natural number } l\\ 0 & \text{otherwise.}\end{cases}$$

The following lemma is the Siegel-Walfisz prime number theorem for arithmetic progressions [D, p.133].

Lemma 5.2 : *Suppose* $1 \le D \le (\log N)^u$, *and* $(m,D) = 1$. *Then for some* $C > 0$,

$$\sum_{\substack{1\le n\le N\\ n\equiv m \pmod{D}}}\Lambda(n) \;=\; \frac{N}{\phi(D)} \;+\; o(Ne^{-C(\log\, N)^{\frac{1}{2}}}).$$

Lemma 5.3 : *For all* $\chi_t \in \hat{\mathbb{Z}}_{a_1} \times \ldots \times \hat{\mathbb{Z}}_{a_s}$*, we have*

$$G^{(1)}_{D,\eta}(\chi_t) \;=\; \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \chi_t(\rho(n)) \tag{1}$$

and

$$G^{(2)}_{D,\eta}(\chi_t) \;=\; \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \chi_t(\rho(p_n)), \tag{2}$$

for appropriate choices of D *and* η*.*

P r o o f. We first prove (1). Note that

$$\chi_t(\rho(n)) \;=\; \chi_t(\alpha_0 \;+\; \alpha_1 n \;+\; \cdots \;+\; \alpha_k n^k),$$

where $\alpha_0, \ldots, \alpha_k$ are the coefficients of the polynomial ρ and are rationals. This is

$$= \; \chi_t(\alpha_0) \prod_{j=1}^{k} (\chi_t(\alpha_j))^{n^j}$$

$$= \; \chi_t(\alpha_0) \prod_{j=1}^{k} (\chi_{\frac{l_1}{A_{r_1,1}}}(\alpha_j) \ldots \chi_{\frac{l_s}{A_{r_s,s}}}(\alpha_j))^{n^j}$$

where t is the character $\left(\frac{l_1}{A_{r_1,1}}, \ldots, \frac{l_s}{A_{r_s,s}}\right)$. If we now write $\alpha_j = \frac{a_i}{q_i}$ $(j = 0, 1, \ldots, s)$, this is

$$= \chi_t(\alpha_0) \prod_{j=1}^{k} \left(e^{2\pi i \frac{a_j l_j}{A_{r_j,j} q_j}} \right)^{n^j} = \chi_t(\alpha_0) e^{2\pi i \frac{\eta(n)}{D}},$$

where η is a polynomial of degree k with coefficients in $\mathbb{Z}$ and D the least common multiple of the integers $q_i A_{r_i,i}$ $(i = 1, 2, \ldots, s)$. Evidently $e^{2\pi i \frac{\eta(n)}{D}}$ depends only on the residue class n belongs to modulo D. This means that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_t(\rho(n)) \;=\; \frac{\chi_t(\alpha_0)}{D_r} \sum_{m=1}^{D} e^{2\pi i \frac{\eta(m)}{D}},$$

proving (1).

We now show (2). Let π_N denote the number of primes less than N and let p denote a prime parameter. Arguing as in (1)

$$\frac{1}{\pi_N} \sum_{1\le p\le N} \chi_t(\rho(p)) \;=\; \frac{\chi_t(\alpha_0)}{\pi_N} \sum_{1\le p\le N} e^{2\pi i \frac{\eta(p)}{D}}. \tag{3}$$

Using partial summation we readily see that

$$(4) \qquad \frac{1}{\pi_N}\sum_{1\le p\le N} e^{2\pi i\frac{\eta(p)}{D}} \;=\; \frac{1}{N}\sum_{1\le n\le N}\Lambda(n)e^{2\pi i\frac{\eta(n)}{D}} \;+\; O((\log\ N)^{-1}).$$

Now note that

$$\sum_{n=1}^{N}\Lambda(n)e^{2\pi i\frac{\gamma(n)}{D}} \;=\; \left(\sum_{\substack{m=1\\(m,D)=1}}^{D} e^{2\pi i\frac{\eta(m)}{D}}\right)\left(\sum_{\substack{1\le n\le N\\ n\equiv m\ (mod\ D)}}\Lambda(n)\right)$$
$$+\,O\left(\sum_{p^l\le N;p|D}\Lambda(p^l)e^{2\pi i\frac{\eta(p^l)}{D}}\right).$$

Using the fact that the third sum on the right is $O((\log\ N)(\text{loglog}\ N))$ and Lemma 3
(5)

$$\frac{1}{N}\sum_{n=1}^{N}\Lambda(n)e^{2\pi i\frac{\gamma(n)}{D}} \;=\; \left(\frac{1}{\phi(D)}\sum_{\substack{m=1\\(m,D)=1}}^{D} e^{2\pi i\frac{\gamma(m)}{D}}\right) \;+\; O\left(\frac{(\log\ N)(\text{loglog}\ N)}{N}\right).$$

Combining (3), (4) and (5) and letting N tend to infinity (4) is proved.

□

We also need the following variant of Wiener condition for the continuity of a measure on the group $\mathbb{Z}_{\mathbf{a}_1}\times\ldots\times\mathbb{Z}_{\mathbf{a}_s}$ [BM]. Let $Y_r = Z_{\mathbf{r}}$ for $\mathbf{r} = (r,\ldots,r)$.

Lemma 5.4 : *If (c_t) and (Y_r) are as as defined in Lemma 4.1, then if*

$$\lim_{r\to\infty}\frac{1}{|Yr|}\sum_{t\in Y_r}|c_t|^2 \;=\; 0,$$

the sequence $(y_n)_{n=0}^{\infty}$ is asymptotically distributed with respect to a non-atomic measure on $\mathbb{Z}_{\mathbf{a}_1}\times\ldots\times\mathbb{Z}_{\mathbf{a}_s}$.

We need Weyl's inequality [Va].

Lemma 5.5 : *Suppose that the integers a and q are coprime and that*

$$\left|\alpha_k \;-\; \frac{a}{q}\right| \;\le\; q^{-2}$$

where

$$\phi(x) \;=\; \alpha_k x^k \;+\; \cdots\; + \alpha_1 x \;+\; \alpha_0.$$

Then

$$\sum_{x=1}^{Q} e^{2\pi i\phi(x)} \ll Q^{1+\epsilon}(q^{-1} \ + \ Q^{-1} \ + \ qQ^{-k})^{1/K},$$

where $K \ = \ 2^{k-1}$.

We now complete the proof the continuity of the distribution measures of the sequences $k_j = \rho(n)$ and $\rho(p_j)$ $(j = 1, 2 \ldots)$. As a consequence of Lemma 5.5 we see that there exists a $\delta \ > \ 0$ such that $|G^{(1)}| \ll D^{-\delta}$. Also

$$\sum_{\substack{1 \le m \le q \\ (m,q)=1}} e^{2\pi i a \eta(m) q^{-1}} \ = \ \sum_{1 \le m \le q} e^{2\pi i a \eta(m) q^{-1}} \ - \sum_{p|q} \sum_{\substack{1 \le m \le q \\ p|m}} e^{2\pi i a \eta(m) q^{-1}}.$$

Hence we also have $|G^{(2)}| \ll D^{-\delta}$ for a possibly different δ. This means that in either case

$$\frac{1}{|Z_r|} \sum_{t \in Z_r} |c_t|^2 \ \le \ \frac{1}{|Z_r|} \sum_{s=0}^{r} |Z_s|^{1-2\delta},$$

which tends to zero as r tends to infinity because $|Z_r| \ \ge \ 2^r$ $(r = 1, 2, ...)$.

Thus the analogue of the Wiener continuity criterion on $\mathbb{Z}_{\mathbf{a}_1} \times \ldots \times \mathbb{Z}_{\mathbf{a}_s}$ [**BM**] shows that the sequences $(\rho(n), \ldots, \rho(n))_{\{n \ge 0\}}$ and $(\rho(p_n), \ldots, \rho(p_n))_{\{n \ge 0\}}$ are asymptotically distributed with respect to a non-atomic measure on the group $\mathbb{Z}_{\mathbf{a}_1} \times \ldots \times \mathbb{Z}_{\mathbf{a}_s}$. Using the properties of the Monna map this means that the sequences $(\phi_{b_1}(\rho(n)), \ldots, \phi_{b_s}(\rho(n)))_{n \ge 0}$ and $(\phi_{b_1}(\rho(p_n)), \ldots, \phi_{b_s}(\rho(p_n)))_{n \ge 0}$ are asymptotically distributed on $[0, 1)^s$ also with respect to a non-atomic measure. Of course this does not preclude the possibility that this measure is Lebesgue measure, in which case the sequences in question would be uniformly distributed. In fact this is not the case in general and neither sequence is uniformly distributed on $[0, 1)^s$ in general. To see this, set $\rho(n) = n^2$ and choose choose l such that the character $l/a_0 \ldots a_s$, is of the form $\frac{a}{q}$ for a prime q with $a \neq 0$. In this case $G^{(1)}_{\frac{l}{b^s}}$ is of the form $C \sum_{r=1}^{q} e^{\frac{2\pi i a r^2}{q}}$ for a non-zero constant C and , as is well known $\left| \sum_{r=1}^{q} e^{\frac{2\pi i a r^2}{q}} \right| \ = \ q^{\frac{1}{2}}$ [**Ap**]. This means it is non-zero and the Fourier transform of the distribution measure of the squares and hence this can't be Haar measre. This means $(\phi_{b_1}(n^2))_{n \ge 0}$ can't be uniformly distributed on $[0, 1)$. Unfortunately dealing with polynomials more general than $\rho(n) = n^2$ or $s > 1$, is a good deal more complex and is as yet unresolved. This is because it relies on lower bounds for exponential sums of the form $\left| \sum_{r=1}^{q} e^{\frac{2\pi i a \rho(r)}{q}} \right|$, which seem a serious undertaking and yet to appear in the literature.

On the other hand, the defining property of Hartman uniformly distributed sequences $(k_j)_{j \ge 0}$, readily implies that $(\phi_{b_1}(k_j), \ldots, \phi_{b_s}(k_j))_{j \ge 0}$ is uniformly distributed on $[0,1)^s$, even without the assumption that the sequence $(k_j)_{j\geq 0}$ is required to be L^∞-good universal.

6. Folner uniform distribution of Halton multi-sequences

Suppose the real numbers $\beta_1,\ldots,\beta_s > 1$ are as in Theorem 1. We will call a series of neighbourhoods of $\mathbb{N}^s$ denoted $(N_t)_{t\geq 1}$ averaging if

(i)
$$N_{t_1} \subseteq N_{t_2}$$
if $t_1 \leq t_2$;

(ii) for each $h = (h_1,\ldots,h_s) \in \mathbb{N}^s$ we set
$$h + N_t = \{(h_1+n_1,\ldots,h_s+n_t) : (n_1,\ldots,n_s) \in \mathbb{N}^s\},$$
then
$$\lim_{t\to\infty} \frac{\#\{(h+N_t)\Delta N_t\}}{\#N_t} = 0;$$
(Here of course Δ denotes the symmetric difference of set.)

(iii) let
$$N_t - N_t = \{x-y : x,y \in N_t\},$$
then there exists a positive constant $K>0$ (possibly dependent on $(N_t)_{t\geq 1}$ and s but not on t) such that
$$\#(N_t - N_t) \leq K\#N_t,$$
for each $t \geq 0$.

We say a multi-sequence $(x_{n_1,\ldots,n_s})_{(n_1,\ldots,n_s)\in\mathbb{N}^s}$ is Folner uniformly distributed on $[0,1)^s$ if for each family of neighbourhood $(N_t)_{t\geq 1}$ satisfying (i) and (ii) and each "box" $B \subseteq [0,1)^s$ of the form
$$B = [a_1,b_1) \times \ldots \times [a_s,b_s)$$
with $a_i < b_i$ $(i=1,2,\ldots,s)$, we have
$$\lim_{t\to\infty} \left| \frac{1}{N_t} \sum_{(n_1,\ldots,n_s)\in N_t} \chi_B(x_{(n_1,\ldots,n_s)}) - |B| \right| = 0.$$

Theorem 6.1. *Suppose $\beta_1,\ldots,\beta_s > 1$ are as in Theorem 1. Suppose the system of neighbourhoods $(N_t)_{t\geq 0}$ contained in $\mathbb{N}^s$ is averaging. Then the multi-sequence $((\phi_{\beta_1}(n_1),\ldots,\phi_{\beta_s}(n_s)))_{(n_1,\ldots,n_s)\in\mathbb{N}^s}$ is Folner uniformly distributed in $[0,1)^s$.*

To prove this theorem we need the following three lemmas.

Lemma 6.2. *Let $(X, \mathcal{B}, \lambda)$ be a probability space and suppose $T_i : X \to X$ $(i = 1, 2, \ldots)$ are measurable measure-preserving maps of X that commute, at least one of which is ergodic, i.e. if $T_i^{-1}(B) = \{x \in X : T_i x \in B\}$, then $T^{-1}B \in \mathcal{B}$ if $B \in \mathcal{B}$, that $\lambda(T_i^{-1}(B)) = \lambda(B)$ for all $i = 1, 2, \ldots, s)$ and $\lambda(T_i(B)\Delta B) = 0$ means $\lambda(B)$ is either 0 or 1 for one $i \in \{1, 2, \ldots, s\}$. Suppose the collection of neighbourboods $(N_t)_{t\geq 0}$ contained in $\mathbb{N}^s$ is averaging. Then if $f \in L^1(X, \mathcal{B}, \lambda)$,*

$$\lim_{t\to\infty} \frac{1}{\# N_t} \sum_{(k_1,\ldots,k_s)} f(T_1^{k_1} \ldots T_s^{k_s} x) = \int_X f d\lambda,$$

λ almost everywhere.

Lemma 6.3. *Suppose for each $t \in \mathbb{Z}_+$ that $N_t = N_t^* \cap \mathbb{Z}^s$, where N_t^* denotes a bounded subset of $\mathbb{Z}^s$. Suppose there exists a positive constant K (possibly dependent on s and $(N_t)_{t\to\infty}$) and an unbounded sequence of numbers $(\rho_s)_{s\geq 0}$ and a sequence of points $(y_t)_{t\geq 0}$ in $\mathbb{R}^s$ such that for each t*

$$B(y_t, \rho_t) \subseteq N_t^* \subseteq N(y_t, K\rho_t).$$

Then $(N_t)_{t\geq 0}$ is averaging.

By appropriate choise of $(\rho_t)_{t\geq 0}$ and $(N_t)_{t\geq 0}$ this lemma can be used to give us broad class of quite natural families of neighbourhoods of convex neighbourhoods, contained in $\mathbb{N}^s$.

Lemma 6.4. *Suppose $(X, \mathcal{B}, \mu)$, $\{T_1, \ldots, T_s\}$ and $(N_t)_{t\geq 0}$ as in Lemma 6.2, then the following are equivalent :*

(i) If $f \in C(X)$ (the space of continuous functions on X) then

$$\lim_{t\to\infty} \frac{1}{\# N_t} \sum_{(k_1,\ldots,k_s)} f(T_1^{k_1} \ldots T_s^{k_s} x) = \int_X f d\lambda,$$

for all $x \in X$;

(ii) If $f \in C(X)$ (the space of continuous functions on X) then there exist a constant C_f such that

$$\lim_{t\to\infty} \frac{1}{\# N_t} \sum_{(k_1,\ldots,k_s)} f(T_1^{k_1} \ldots T_s^{k_s} x) = C_f,$$

for all $x \in X$;

(iii) If $f \in C(X)$ (the space of continuous functions on X) then there exist a constant C_f such that

$$\lim_{t\to\infty} \frac{1}{\#N_t} \sum_{(k_1,\ldots,k_s)} f(T_1^{k_1}\ldots T_s^{k_s}x) = C_f,$$

uniformly for all $x \in X$;

and

(iv) One of the dynamical systems $(X, \mathcal{B}, \mu, T_i)$ $(i = 1, 2, \ldots, s)$ is uniquely ergodic.

P r o o f. Only (iv) implying (iii) is not a routine modification of the corresponding argument in Lemma 2.2. To prove this we argue as follows. Set

$$S_t(f,x) = \frac{1}{\#N_t} \sum_{(k_1,\ldots,k_s)} f(T_1^{k_1}\ldots T_s^{k_s}x). \qquad (t = 1, 2, \ldots)$$

If (iii) were true we must have $C_f = \int_X f d\lambda$. Assume for the sake of contradiction that there exist $\epsilon > 0$, continuous $g \in C(X)$ and $(x_n)_{n\geq 0}$ such that if given $N \in \mathbb{N}$ there exists $n > N$ such that

$$\left| S_n(g, x_n) - \int_X g d\lambda \right| > \epsilon.$$

Let

$$\mu_t = \frac{1}{\#N_t} \sum_{(k_1,\ldots,k_s)\in N_t} \delta_{T_1^{k_1}\ldots T_s^{k_s}x_n}, \qquad (t = 1, 2, \ldots)$$

where δ_y refers to the delta function at y. As $M(X)$ is compact the sequences $(\mu_t)_{t\geq 0}$ has a subsequence with a limit μ_∞ in $M(X)$. We must show μ_∞ is T - invariant. Then the assumed unique ergodicity of one of the maps T_i $(i = 1, 2, \ldots)$ implies μ_∞ and μ are different. This implies (iii) as required. Let $f \in C(X)$. Then for each i in $\{1, 2, \ldots, s\}$ we have

$$\left| \int_X f T_i d\mu_\infty - \int_X f d\mu_\infty \right|$$

$$= \lim_{t\to\infty} \left| \frac{1}{\#N_t} \left(\sum_{(k_1,\ldots,k_s)\in N_{t,i}^+} - \sum_{(k_1,\ldots,k_s)\in N_t} \right) \int_X f(T_1^{k_1}\ldots T_s^{k_s}x) \right|$$

where $N_{t,i}^+$ denote the set N_t with the i^{th} entry of each element shifted up by 1. Now because of property (ii) of averaging we know that $\#N_{t,i}^+ - \#N_t = o(\#N_t)$

as t tends to infinity. This means μ_∞ is T_i invariant for each $i \in \{1, 2, \ldots, s\}$. Thus μ_∞ and μ are different. This means (iv) implies (iii) as required. $\square$

7. Moving averages

For the theorem which we wish to prove we need to restrict attention to certain classes of rectangles $R = (R_k)_{k=1}^\infty$. In particular suppose $n_k = (n_{1,k}, \cdots, n_{d,k})$ is the south west corner of the rectangle R_k. Suppose also that the rectangle R_k has sides $(l_{1,k}, \cdots, l_{d,k}) \in \mathbf{N}^d$. We also assume $l_{i,k} \leq l_{i,k+1}$ $(1 \leq i \leq d)$. Assume $\alpha \in (0, \infty)$. Set

$$\Omega_\alpha^i = \{(z, s) \in \mathbf{Z} \times \mathbf{Z}_+ : |z - n_{i,k}| \leq \alpha(s - l_{i,k}) \text{ for some } k\}$$

and set $\Omega_\alpha^i(\lambda) = \{z : (z, \lambda) \in \Omega_\alpha^i\}$ $(\lambda \in \mathbf{Z}_+)$. We call R *good* if there exists $A = A_\alpha \in (0, \infty)$ such that $|\Omega_\alpha^i(\lambda)| \leq A\lambda$ $(1 \leq i \leq d)$.

Let $(T_i)_{i=1}^k$ be a set of commuting maps of L^∞ induced by commuting measurable measure-peserving transformations $(\tau_i)_{i=1}^k$ of the probability space (X, β, μ) by setting $T_i f(x) = f(\tau_i x)$ $(i \leq i \leq d)$. For the multi-index $j = (j_1, \cdots, j_d) \in \mathbf{N}^d$ write $T^j f(x) = T_1^{j_1} \cdots T_d^{j_d} f(x)$, write $S_k(R, f)(x) = \sum_{j \in R_k} T^j f(x)$ and write $A_k(R, f)(x) = \frac{1}{|R_k|} S_k(R, f)(x)$. The following is proved in [JO].

Theorem 7.1. *If $f \in L^1(X, \mathcal{B}, \mu)$ and R is a good sequence of rectangles, then $f^*(x) = \lim_{k\to\infty} A_k(R, f)(x)$ exists almost everywhere.*

To the sequence of rectangles R we associate the sequence of rectangles $R^{i,+} = (R_k^{i,+})_{k=1}^\infty$ $(1 \leq i \leq d)$ where for in each rectangle the element $(n_1, \ldots, n_d)$ is replaced by the element $(n_1, \ldots, n_{i-1}, n_i + 1, n_{i+1}, \ldots, n_d)$. Notice that

$$A_k(R, f)(x) - A_k(R, T_i f)(x) = \frac{1}{|R_k|}(S_k(R, f)(x) - S_k(R^{i,+}, f)(x)).$$

Also if Δ denotes symmetric difference, self evidently $\lim_{k\to\infty} \frac{|R_k \Delta R_k^{i,+}|}{|R_k|} = 0$. This means $T_i f^*(x) = f^*(x)$ $(1 \leq i \leq d)$ and so $T^j f^*(x) = f^*(x)$ for $j \in \mathbf{N}^d$. This means that if any of the maps $\{T_1, \ldots, T_d\}$ is ergodic then $f^*(x) = \int_X f(x) d\mu$. We have the following result whose proof is similar to that of Lemma 6.4 and is hence forgone.

Lemma 7.2. *Suppose $(X, \mathcal{B}, \mu)$, $\{T_1, \ldots, T_s\}$ and $(N_t)_{t \geq 0}$ as in Lemma 6.2, then the following are equivalent :*

(i) If $f \in C(X)$ (the space of continuous functions on X) then

$$\lim_{k \to \infty} A_k(R, f)(x) = \int_X f d\lambda,$$

for all $x \in X$;

(ii) If $f \in C(X)$ (the space of continuous functions on X) then there exist a constant C_f such that

$$\lim_{k \to \infty} A_k(R, f)(x) = C_f,$$

for all $x \in X$;

(iii) If $f \in C(X)$ (the space of continuous functions on X) then there exist a constant C_f such that

$$\lim_{k \to \infty} A_k(R, f)(x) = C_f,$$

uniformly for all $x \in X$;

and

(iv) One of the dynamical systems $(X, \mathcal{B}, \mu, T_i)$ $(i = 1, 2, \ldots, s)$ is uniquely ergodic.

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